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ANALYTIC DEVELOPMENT OF THE FLIGHT PATH EQUATIONS FOR MAXIMUM RANGE BY THE METHODS OF VARIATIONAL CALCULUS

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ANALYTIC DEVELOPMENT OF THE FLIGHT PATH EQUATIONS FOR MAXIMUM RANGE BY THE METHODS OF VARIATIONAL CALCULUS

by
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ABBREVIATIONS AND SYMBOLS TO BE USED IN THIS REPORT

AR	Wing aspect ratio
C_D	Aircraft total drag coefficient
C_{D_i}	Induced drag coefficient
C_{D_o}	Drag coefficient at zero lift
C_T	Thrust specific fuel consumption
CF	Centrifugal force
D	Total aircraft drag
d	Total differential operator
∂	Partial differential operator
D_i	Induced drag
D_o	Drag at zero lift
e	Airplane efficiency factor
F	Total effective thrust
F^*	Thrust due to momentum change of air
f	Effective thrust available
g	Gravitational acceleration constant (32.17 ft/sec^2)
h	Altitude
L	Aircraft lift
m	Mass of air and fuel
M	Mach number
q	Dynamic pressure ($q = \frac{1}{2} \rho V^2$)
S	Wing surface area
t	Time
u	Engine exhaust velocity relative to the origin
u_e	Exhaust velocity relative to the body ($V - u$)

V	Velocity
V_c	Speed of sound
$\dot{V}$	Velocity derivative with respect to time
W	Total weight of body or aircraft
${}_1W_2$	Weight of fuel expanded from point 1 to 2
x	Horizontal distance
γ	Flight path angle relative to the x-axis
$\dot{\gamma}$	Flight path angle derivative with respect to time
ρ	Air density
ω	Angular velocity in pitch
C_L	Aircraft total lift coefficient

SUMMARY

In this report the principles of variational calculus are applied to derive the equations for a flight path resulting in either maximum range for a given fuel expenditure or minimum fuel expenditure for a given range. The equations so determined permit solution of the problem without making the normal restrictive assumptions of lift equals weight and thrust equals drag. This permits analytical treatment of flight attitudes ranging from level flight to vertical flight and accelerations up to structural or pilot limitations. With such a general treatment of aircraft motion, it will be possible to compute schedules for maximum range flight paths commencing at any physically attainable combination of altitude, speed, and distance.

The equations obtained for these extremal flight paths are first and second order nonlinear differential equations. Numerical solutions of this system are shown to be possible using stepwise integration techniques.

Classical parameters for maximum range flight are readily derived from the path equations by specifying level flight only, indicating that classical range equations are restrictive applications of the more general equations derived in this thesis.

This investigation was conducted in partial fulfillment of the requirements for the degree of Master of Science in Engineering by Lieutenant Commander Charles W. Fritz, USN, and Lieutenant John [illegible]. Tierney, USN, during the 1954 - 1955 academic year at the Forrestal Research Center, Princeton University, Princeton, New Jersey.

INTRODUCTION

The problem of determining the maximum range schedule of an aircraft with a specified fuel load has received very little attention since the Breguet developments and the more recent work on cruise charts for jet type aircraft. These methods are very restrictive in that they apply principally to low performance aircraft. Any improvement, either in aircraft design or in flight path schedules, which would increase the range for a given amount of fuel burned in an aircraft would be most enthusiastically received by designers, tacticians, and anyone concerned with aviation. In the following pages a method for determination of flight path schedules for maximum range will be developed which, the authors feel, can produce significant improvements over any method now in use.

Man's intuitive sense will tell him that for all possible paths in space, there will be one particular path which will carry him furthest for a given fuel expenditure. This same sense tells us that for a given distance traveled only one path will yield the minimum fuel consumption. Restricting the general case to the operating dimensions and capabilities of modern aircraft, it can be seen that for every flight path of variable altitude and speed, there will be one particular schedule of altitude and speed which will give the greatest range for a given fuel consumption. There will also be one flight schedule which will give the minimum fuel consumption for a given range. Fig. 1 illustrates this graphically. The most general aircraft problem will have loosely defined initial conditions. The path for minimum fuel could be started from any combination of inital speed

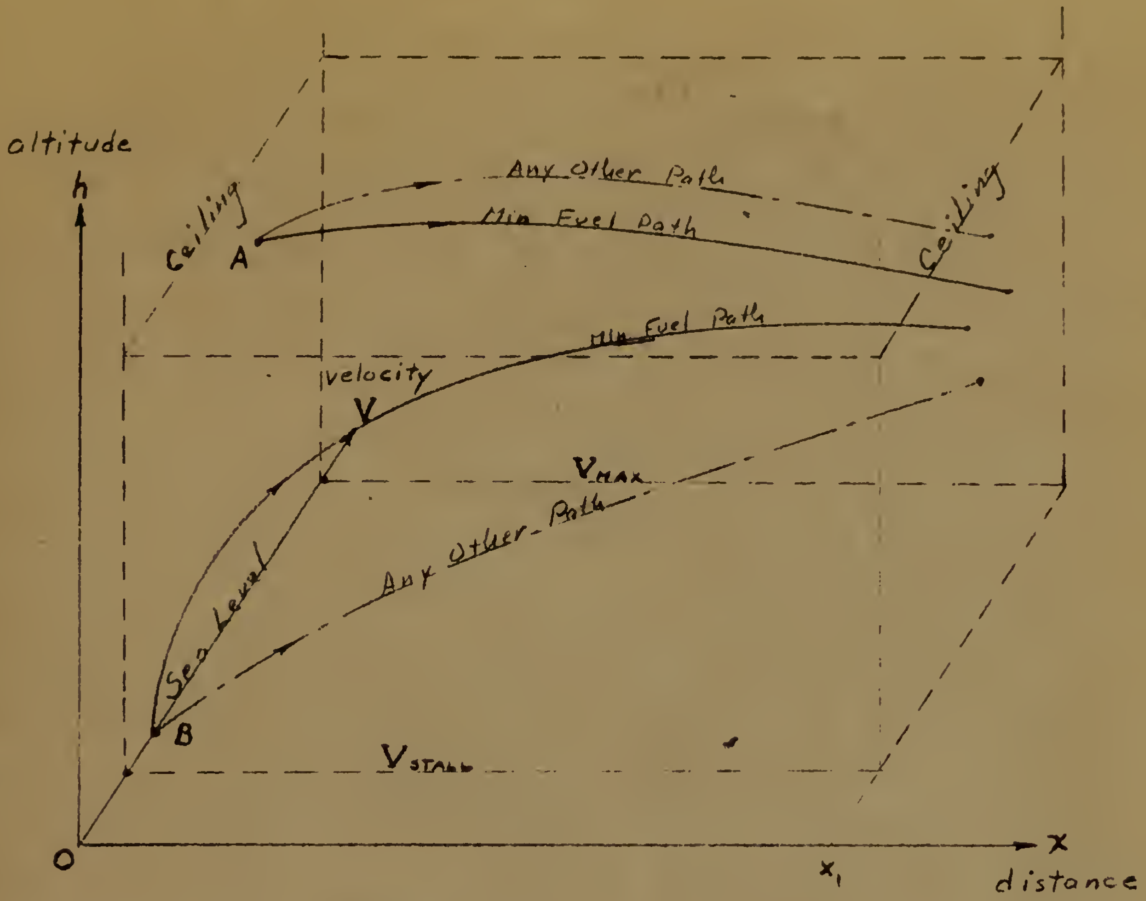
altitude
h
Ceiling
A
Any Other Path
Min Fuel Path
Ceiling
Min Fuel Path
velocity
V
V_MAX
Any Other Path
Sea Level
B
V_STALL
O
x
x₁
distance

Fig. 1

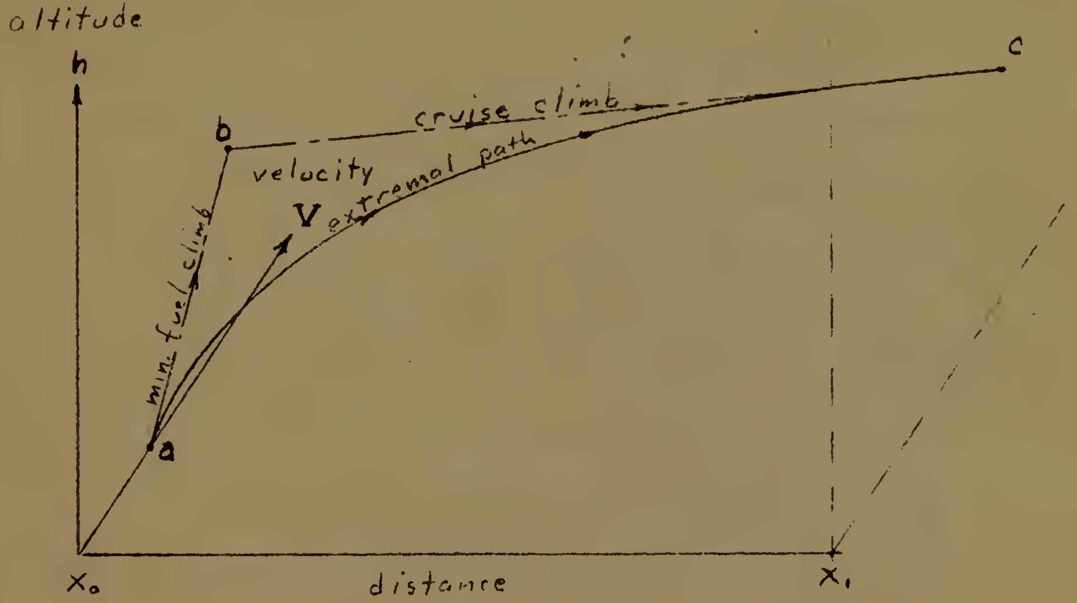
altitude
h
b
cruise climb
c
velocity
V
extremal path
min fuel climb
a
x₀
distance
x₁

Fig. 2

and altitude, such as pt. A on fig. 1, and flown a given distance x_1. The schedule of altitude and speed would be determined by the interrelations of the variables of the problem bounded by the capabilities of the particular aircraft. The problem could just as well be started form the instant of take-off, say pt. B, and thus cover the entire flight from the take-off until over x_1.

Commencing a maximum range or minimum fuel schedule from take-off is a completely new concept. Classical range solutions do not attempt to solve for an optimum flight path from point of take-off to optimum cruising altitude, but rather specify, for turbo-jets, a nebulous "minimum fuel climb" to the highest altitude at which speed for $(C_L^{1/2}/C_D)_{max}$ is equal to V_{max}. The aircraft then cruises at that altitude, slowly climbing as the weight is decreased, thus increasing the altitude at which the speed for maximum $C_L^{1/2}/C_D$ is equal to the maximum speed.

This "minimum fuel climb" will, if properly determined, achieve a certain altitude with a minimum fuel expenditure, but this is not the primary concern. The problem is to travel a distance x_1 with minimum fuel consumed. If the variables of the problem call for a climb to a specified altitude, then that climb should be executed to give maximum distance for a given fuel expenditure to get to that altitude. As long as the flight path stays within the limitations of the aircraft, however, there should be a single path with no discontinuities for minimum fuel path for a given distance. Thus looking at fig. 2, it can be seen that to travel from pt. *a* to some pt. *c* a distance x_1 away there must be some minimum path *ac* which uses

less fuel than the path abc which is composed of two parts, even though ab and bc are separate paths.

Classical range equations, moreover, are not generally applicable or accurate for present day high performance aircraft. Classical range performance is based on assuming that the flight path angle (γ) is small such that its cosine is considered one, (this assumes that $L = W$) and that flight path accelerations along the path were ignored (i.e. Thrust = Drag). From a review of the literature on the subject, it was determined that everything written to date on the range of aircraft has been based on these simplifying but very restrictive assumptions. With modern high performance aircraft, flight path angles up to and including 90 degrees are possible and flight path accelerations can easily exceed 1.0 g. In fact, in considering the general problem of powered flight (rockets, missels, VTO, etc.), the convenient and conventional condition of steady level flight is almost the exception instead of the rule.

It is the intent of this work to indicate methods based on the principals of variational calculus for the determination of schedules for minimum fuel or maximum range flights for all types of powered flights. It is intended, more specifically, to develop equations for the minimum fuel paths of turbo-jet type aircraft. This differs from the classic problem in that the schedule begins at take-off and minimizes the fuel expenditure from that point to a distance x_1, taking into consideration possible large values of flight path angle and considerable flight path accelerations. In other words, the problem considered here is the broad general problem. It uses accurate, realistic conditions and minimizes fuel consumption for the entire

flight path from "wheels up" to "over target".

It is hardly necessary to discuss the importance of reduced fuel consumption since its advantages are so readily apparent, but to help keep these advantages in mind, a few of them will be listed:

(1) Increased useful payload (bombs, cargo, passengers, etc)

(2) Greater range or combat radius with the same payload

(3) More economical operation overall

(4) Complete specification of a flight schedule which could be incorporated into an automatic control for a manned or pilotless aircraft.

The adaptability of these and other advantages to all phases of aviation is also readily apparent. The long range bomber, the short range fighter, the commercial cargo and passenger carrier, the pilotless aircraft, and practically all other vehicles traveling through the air could adopt flight schedules determined in this manner to increase their effectiveness and their efficiency.

A specific application for which this work could be immediately useful is the determination of entry and escape paths for a supersonic bombing mission.. Those parts of the entry and escape path not determined by other tactical considerations would most probably be flown at full power including afterburner and following a path to give best fuel economy at maximum power. The bounded thrust path equations developed later in this thesis apply directly to this problem.

In this and many other applications, Mach effects must be considered. For simplicity, however, Mach effects have been omitted from the derivations that follow, but have been treated separately in Appendix B.

PART TWO: DEVELOPMENT OF EQUATIONS

The range of an aircraft is, as we have hypothesised, a function of its flight path. Its flight path, in turn, depends on variables such as amount of fuel, velocity along path, flight path angle, accelerations along and normal to the path, and thrust specific fuel consumption. These variables are, in turn, functions of each other, drag, thrust, lift, weight, and air density. Relating these variables among sufficient equations to solve for a minimum fuel path is the problem of this thesis.

A) EQUATIONS OF MOTION

The equations of motion pertinent to the problem are the force balance equations of a body moving in a vertical plane. This motion will include accelerations along and normal to the flight path, as in maneuvering flight, since these accelerations will, a priori, affect the loads and thrust required, and therefore affect the range (or fuel consumption). The forces acting on a body moving along an arbitrary path in two dimensional space are shown in fig. 3.

If this body moving in space is a conventional aircraft, the forces F_1, F_2, and F_3 can be expressed as:

F_1 The total effective thrust. (F)

F_2 The lift of the moving body. (L)

F_3 The total drag force. (Body drag, D, and inertia, $m\frac{dV}{dt}$)

By equating forces parallel and perpendicular to the flight path to zero, the following equations are obtained:

(1) Forces parallel to flight path

$$\Sigma F_{\parallel} = 0 = F_1 - F_3 - W\sin\gamma \tag{1}$$

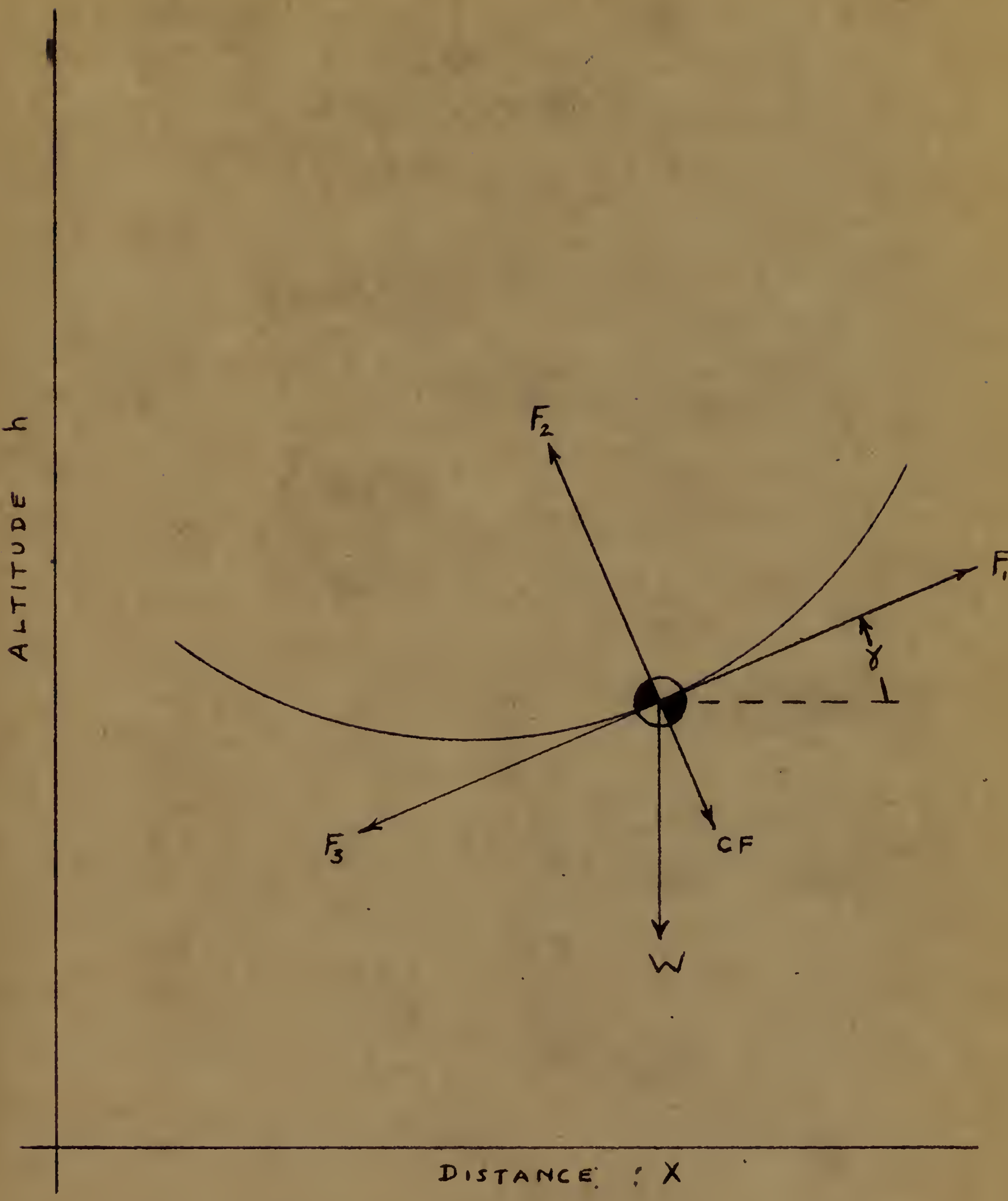

Fig 3

FORCES ACTING ON BODY

or for an aircraft:

$$F^{*} - D - \frac{d\,(mV)}{dt} - W\sin\gamma + u\,\frac{dm}{dt} = 0 \qquad (2)$$

where F, D, W, γ, u, and m were defined previously and:

$$\frac{d(mV)}{dt} = m\,\frac{dV}{dt} + V\,\frac{dm}{dt}$$

But

$$\frac{dm}{dt} = \frac{-C_T F}{g} \qquad (3)$$

since fuel is the only variable mass involved.

Also $\frac{dV}{dt} \equiv \dot{V}$

Now

$$\Sigma F_{\parallel} = F^{*} - D - m\dot{V} + \frac{V C_T F}{g} - W\sin\gamma - \frac{u C_T F}{g} = 0 \qquad (4)$$

Considering the definition of thrust, it can be seen that the total effective thrust acting on the aircraft would be equal to that imparted by the air through the engine as it experiences a change of momentum plus that imparted by the fuel mass leaving in the exhaust due to its momentum change. The thrust due to the leaving fuel is a function of the difference between aircraft velocity and exhaust velocity and is expressed by:

$$(u - V)\,\frac{C_T\,F}{g}$$

Thus

$$F = F^{*} - (u - V)\,\frac{C_T\,F}{g}$$

$$= F^{*} + u_e\,\frac{C_T\,F}{g} \qquad (5)$$

The equation now becomes

$$F - D - \frac{W}{g}\dot{V} - W\sin\gamma = 0 \qquad (6)$$

(2) Forces perpendicular to the flight path

$$\Sigma F_{\perp} = 0 = F_2 - CF - W\cdot\cos\gamma \qquad (7)$$

or for an aircraft

$$L - CF - W\cos\gamma = 0 \qquad (8)$$

But

$$C\,F = m\,V\,\omega$$

and

$$\omega = \frac{d\gamma}{dt} \equiv \dot{\gamma}$$

then

$$C\,F = \frac{W}{g}V\dot{\gamma} \qquad (9)$$

And therefore

$$\Sigma F_{\perp} = L - \frac{W}{g}V\dot{\gamma} - W\cos\gamma = 0 \qquad (10)$$

Equations (6) and (10) above are the general equations of motion pertinent to flight in the vertical plane. These will be modified later for use in the range problem to the extent that derivatives with respect to time will be replaced by parametric derivatives with respect to either weight or distance.

B) EQUATIONS OF MOTION FOR RANGE

Before attempting to maximize the general equations of motion we will further modify them to be more applicable to the problem of,
(1) Maximizing the range, x, for a given fuel expenditure, ${}_1W_2$, or
(2) Minimizing the fuel expenditure, ${}_1W_2$ for a given range, x.

For maximizing range, x, it is convenient to change variables as follows:

Define $d(\ )/dW \quad (\ )'$

Then $\dot{V} = \frac{dV}{dt} = \frac{dV}{dW}\frac{dW}{dt} = V'\frac{dW}{dt}$

But $\frac{dW}{dt} = -C_T F$

Therefore $\dot{V} = -V' C_T F$ (11)

Also

$$\dot{\gamma} = \frac{d\gamma}{dt} = \frac{d\gamma}{dW}\frac{dW}{dt} = \gamma'\frac{dW}{dt}$$

$$\dot{\gamma} = -\gamma' C_T F \qquad (12)$$

Substituting (11) and (12) into (6) and (10) gives:

$$L + \frac{W}{g} V \gamma' C_T F - W\cos\gamma = 0 \qquad (13a)$$

$$F - D + \frac{W}{g} V' C_T F - W\sin\gamma = 0 \qquad (13b)$$

The integral equation relating x and W in simplest form is:

$$x = \int \frac{dx}{dW} dW$$

Now $\frac{dx}{dW} = \frac{dx}{dt}\frac{dt}{dW}$

But $\frac{dx}{dt} = V\cos\gamma$

And from before

$$\frac{dt}{dW} = -\frac{1}{C_T F}$$

so $\frac{dx}{dW} = -\frac{V\cos\gamma}{C_T F}$

And therefore

$$x = \int -\frac{V\cos}{C_T F} dW \qquad (15)$$

From (13b)

$$F = \frac{D + W\sin\gamma}{1 + \frac{WV'C_T}{g}} = \frac{W\cos\gamma - L}{\frac{WV\gamma' C_T}{g}} \qquad (16)$$

Substituting into (15)

$$x = \int \frac{\frac{W}{g} V^2 \gamma' \cos\gamma \; dW}{L - W\cos\gamma} \tag{17}$$

or

$$x = \int - \frac{V\cos\gamma \left(1 + \frac{W}{g} V' C_T\right) dW}{C_T \left(D + W\sin\gamma\right)} \tag{18}$$

(2) For minimizing the fuel used, ${}_1W_2$, the changes of variables found useful are:

Defining $\frac{d(\;)}{dx} \equiv (\;)'$

Then

$$\dot V = \frac{dV}{dt} = \frac{dV}{dx}\frac{dx}{dt} + V' V\cos\gamma \tag{19}$$

And

$$\dot\gamma = \frac{d\gamma}{dt} = \frac{d\gamma}{dx}\frac{dx}{dt} = \gamma' V\cos\gamma \tag{20}$$

Substituting into (6) and (10) gives:

$$L - \frac{W}{g} V^2 \gamma' \cos\gamma - W\cos\gamma = 0 \tag{21a}$$

$$F - D - \frac{W}{g} V' V\cos\gamma - W\sin\gamma = 0 \tag{21b}$$

The integral equation relating x and W in simplest form for minimizing ${}_1W_2$ is:

$$W = \int \frac{dW}{dx} dx$$

As before

$$\frac{dW}{dx} = - \frac{C_T F}{V\cos\gamma} \tag{22}$$

and

$$W = \int - \frac{C_T F}{V\cos\gamma} dx$$

Solving (21b) for F and substituting gives

$$W = \int - C_T \frac{\left(D + \frac{W}{g}V'V\cos\gamma + W\sin\gamma\right)}{V\cos\gamma}dx$$

or

$$W = \int - C_T\left[\frac{D}{V\cos\gamma} + \frac{WV'}{g} + \frac{W\tan\gamma}{V}\right] dx \qquad (23)$$

Now, although equations (18) and (23) relate the variables of the problem, they do not specify the flight path. They are equations for the general motion of a body in the vertical plane, expressed in such a way as to give the relationship between range and fuel expenditure for any flight path. The variables under the integrals are dependent on the path, yet the equations as such do not specify a path. To determine equations for the particular path which will give the greatest range or least fuel expenditure it is necessary to look to the mathematics of maxima and minima. Presenting the problem in more mathematical terms, the unique path desired is that path giving the extremal value of an integral. The problem of finding an estremal value belongs to that branch of mathematics known as "The Calculus of Variations". A brief discussion of this subject follows.

C) THE EULER EQUATIONS OF THE CALCULUS OF VARIATIONS

Let I be a quantity defined as an integral function of several variables as follows:

$$I = \int G(x,y,z, \text{---}, y', z', \text{---},)\, dx$$

where ( )' indicates the first derivative with respect to x. The Calculus of Variations is concerned with finding the extremal values of the quantity so defined. The necessary and sufficient conditions

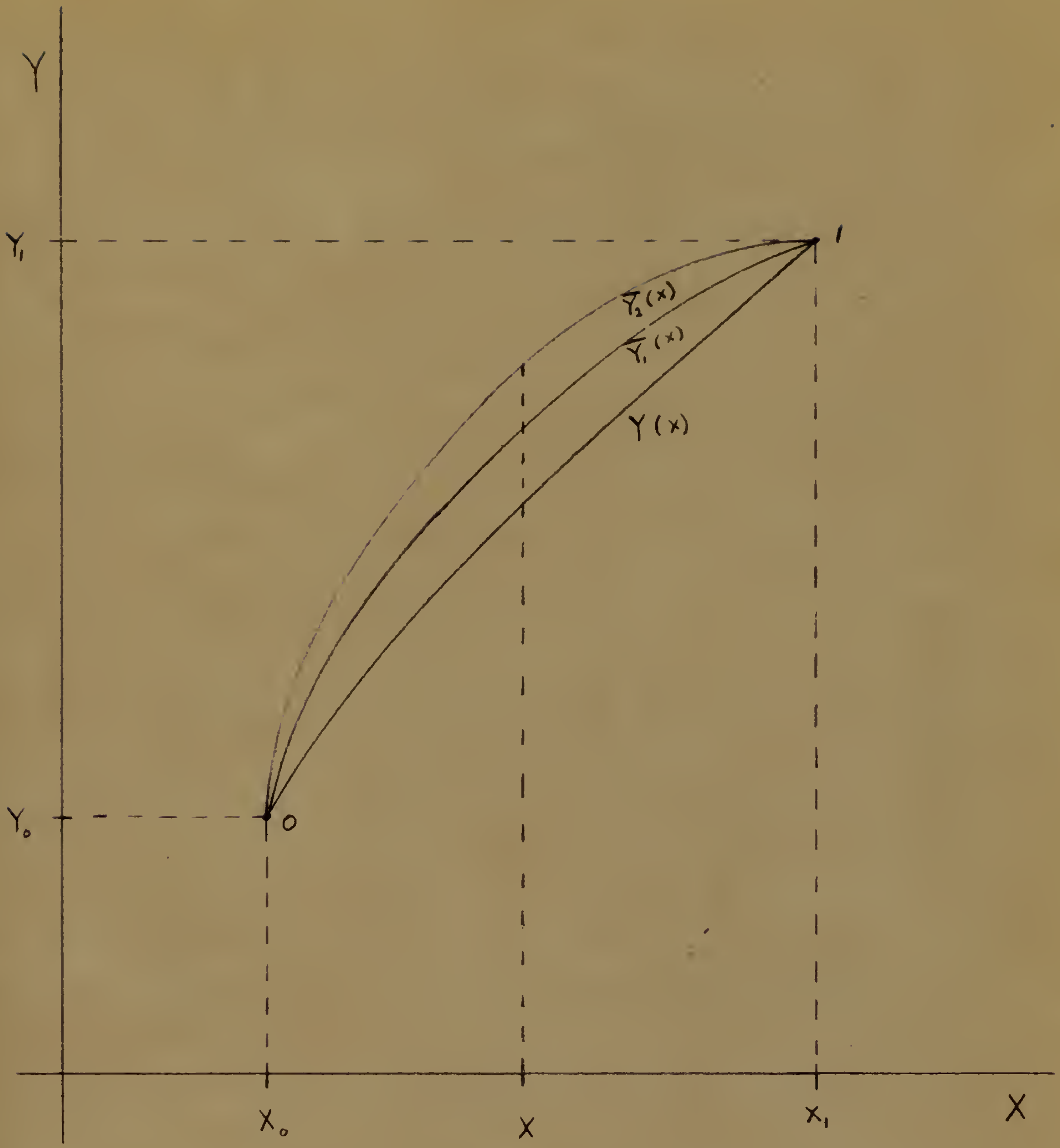

Fig 4

FAMILY OF PATHS FROM POINT "0" TO "1"

$$Y_n'(x) = Y'(x) + \epsilon_n \eta'(x) \qquad (26)$$

With the conditions:

$$\left.\begin{aligned} \bar{Y}_n(x_0) &= Y(x_0) \\ \bar{Y}_n(x_1) &= Y(x_1) \end{aligned}\right\} \quad \text{i.e. } \eta(x) \equiv 0 \text{ at the end points}$$

Where ϵ_n is a small constant quantity and $\eta(x)$ is an arbitrary function of x that is continuous and has continuous first and second derivatives in the range of integration.

$\bar{Y}_n(x)$ therefore defines all possible paths of integration between X_0, Y_0 and X_1, Y_1 so that (24) may be written:

$$I_n = \int_{X_0,Y_0}^{X_1,Y_1} G\left[x, \bar{Y}_n(x), \bar{Y}_n'(x)\right] dx$$

or replacing $\bar{Y}_n$ and $\bar{Y}_n'$ from (25) and (26)

$$I_n = \int_{X_0,Y_0}^{X_1,Y} G\left[X, Y(x) + \epsilon_n \eta(x), Y'(x) + \epsilon_n \eta'(x)\right] dx \qquad (27)$$

It is clearly obvious that I_R differs from I_K only through the value of ϵ and it may be concluded that:

$$I_n = I(\epsilon) = f(\epsilon) \quad \text{only}$$

and since ϵ is not a function of x, the ordinary maxima and minima rule may be applied, i.e.:

$$\left.\frac{dI}{d\epsilon}\right|_{\epsilon \to 0} = 0 \quad \text{for an extremum}$$

and, furthermore, the differentiation may be made under the integral, thus:

$$\frac{dI}{d\epsilon} = \int_{X_0,Y_0}^{X_1,Y_1} \frac{dG}{d\epsilon}\, dx \qquad (28)$$

From (27)

$$\frac{dG}{d\epsilon} = \frac{\partial G}{\partial\left[Y(x) + \epsilon_n \eta(x)\right]} \cdot \frac{d\left[Y(x) + \epsilon_n \eta(x)\right]}{d\epsilon}$$

$$+ \frac{\partial G}{\partial\left[Y'(x) + \epsilon_n \eta'(x)\right]} \cdot \frac{d\left[Y'(x) + \epsilon_n \eta'(x)\right]}{d\epsilon} + \frac{\partial G}{\partial x}\frac{dx}{d\epsilon}$$

Since

$$\frac{dx}{d\epsilon} = \frac{dY}{d\epsilon} = \frac{d\eta}{d\epsilon} = \frac{dY'}{d\epsilon} = \frac{d\eta'}{d\epsilon} = 0$$

then

$$\frac{dG}{d\epsilon} = \frac{\partial G}{\partial\left[Y(x) + \epsilon_n \eta(x)\right]} \cdot \eta(x) + \frac{\partial G}{\partial\left[Y'(x) + \epsilon_n \eta'(x)\right]} \cdot \eta'(x)$$

or

$$\frac{dI}{d\epsilon} = \int_{X_0,Y_0}^{X_1,Y_1} \left[\frac{\partial G}{\partial(Y + \epsilon_n \eta)}\eta + \frac{\partial G}{\partial(Y' + \epsilon_n \eta')}\eta'\right] dx = 0 \qquad (29)$$

and

$$\frac{dI}{d\epsilon} = \int_{X_0,Y_0}^{X_1,Y_1} \frac{\partial G}{\partial(Y + \epsilon_n \eta)}\eta\, dx + \int_{X_0,Y_0}^{X_1,Y_1} \frac{\partial G}{\partial(Y' + \epsilon_n \eta')}\eta' dx = 0$$

From whence

$$\int_{X_0,Y_0}^{X_1,Y_1} \frac{\partial G}{\partial \bar{Y}_n}\eta\, dx = -\int_{X_0,Y_0}^{X_1,Y_1} \frac{\partial G}{\partial \bar{Y}_n'}\eta' dx$$

Integrating the right side by parts and remembering that $\eta(X_0, Y_0) = \eta(X_1, Y_1) = 0$

$$\int_{X_0,Y_0}^{X_1,Y_1} \frac{\partial G}{\partial \bar{Y}_n'}\eta' dx = -\int_{X_0,Y_0}^{X_1,Y_1} \frac{d\left(\frac{\partial G}{\partial \bar{Y}_n'}\right)}{dx}\eta\, dx \qquad (30)$$

Substituting (30) into (29) gives:

$$\frac{dI}{d\epsilon} = \int_{X_0,Y_0}^{X_1,Y_1} \left[\frac{\partial G}{\partial \bar{Y}_n} - \frac{d}{dx}\frac{\partial G}{\partial \bar{Y}_n'}\right]\eta\, dx = 0 \qquad (31)$$

Evaluating (31) for $\epsilon = 0$ to find the extremum:

$$\int_{X_0,Y_0}^{X_1,Y_1} \left[\frac{\partial G}{\partial Y} - \frac{d}{dx}\frac{\partial G}{\partial Y'}\right]\eta\, dx = 0$$

But this is true only if

$$\left[\frac{\partial G}{\partial Y} - \frac{d}{dx}\left(\frac{\partial G}{\partial Y'}\right)\right]\eta = 0$$

throughout the range of integration. Since η is completely arbitrary at every point except the end points, it necessarily follows that for an extremum:

$$\frac{\partial G}{\partial Y} - \frac{d}{dx}\left(\frac{\partial G}{\partial Y'}\right) = 0 \qquad (32)$$

This equation is called the Euler equation. It must be satisfied if the integral is to have an extremum value and it is through this satisfaction that the path, Y(X), is found.

Proceeding to the more general problem of several variables, the integral must satisgy the definitions as before and:

$$I = \int_{X_0, Y_0, Z_0, \cdots}^{X_1, Y_1, Z_1, \cdots} G\left[X, Y, Z, ---, Y', Z', ----\right] dx$$

Proceeding as before, we arrive by an analagous developement at the Euler equations:

$$\left.\begin{aligned} &\frac{\partial G}{\partial Y} - \frac{d}{dx}\left(\frac{\partial G}{\partial Y'}\right) = 0 \\ &\frac{\partial G}{\partial Z} - \frac{d}{dx}\left(\frac{\partial G}{\partial Z'}\right) = 0 \\ &\frac{\partial G}{\partial (\ )} - \frac{d}{dx}\frac{\partial G}{\partial (\ )} = 0 \quad \text{etc.} \end{aligned}\right\} \qquad (33)$$

These equations (33) must be satisfied simultaneously for an extremal value of the integral, such satisfaction therefore defining the desired path as in the two dimensional case.

It should be remembered here that an extremal value for the integral may be either a maximum or a minimum value. Thus in deter-

mining a solution for equations (33), we must determine by other means whether a particular solution will yield a maximum path or a minimum path. For most types of engineering work this will be intuitively apparent.

D) THE EULER EQUATIONS FOR MAXIMUM RANGE

a.) Minimum Fuel Equations (i.e., x independent variable)

Equation (22) is of the form

$$-{}_1W_2 = I(c) = \int_{x_1}^{x_2} G\,dx$$

where $I(c) = -W$ the quantity (of fuel) to be minimized and $G = \frac{C_T F}{V\cos\gamma}$ the integrand function.

Now:

$$G = G(C_T, V, \gamma, F)$$

$$F = F(D, V', W, V, \gamma)$$

$$D = D(L, V)$$

$$L = L(W, V, \gamma, \gamma')$$

thus

$$G = G(W, V, \gamma, C_T, \gamma', V')$$

Now, C_T, the thrust specific fuel consumption, is a variable, and a very important variable where range is concerned, but treating it as such in the derivation of Euler equations gives rise to extremely complicated relations and to derivatives on C_T that are very difficult to evaluate. We have therefore elected to treat C_T as a stepwise variable constant.

Thus $G = G(W, V, \gamma, V', \gamma')$

And the Euler equations of this function become:

$$1.)\quad \frac{\partial G}{\partial V} - \frac{d}{dx}\left(\frac{\partial G}{\partial V'}\right) = 0$$

$$2.)\quad \frac{\partial G}{\partial \gamma} - \frac{d}{dx}\left(\frac{\partial G}{\partial \gamma'}\right) = 0$$

$$3.)\quad \frac{\partial G}{\partial W} - \frac{d}{dx}\left(\frac{\partial G}{\partial W'}\right) = 0$$

1.) Derivation of the first Euler

$$\frac{\partial G}{\partial V} - \frac{d}{dx}\left(\frac{\partial G}{\partial V'}\right) = 0$$

Now from (22)

$$G = \frac{C_T\, F}{V\cos\gamma}$$

Then

$$\frac{\partial G}{\partial V} = \frac{V\cos\gamma\, C_T \frac{\partial F}{\partial V} - C_T\, F \cos\gamma}{(V\cos\gamma)^2}$$

$$= G\left[\frac{\partial F/\partial V}{F} - \frac{1}{V}\right]$$

And

$$\frac{\partial G}{\partial V'} = \frac{C_T}{V\cos\gamma}\left(\frac{\partial F}{\partial V'}\right)$$

$$\frac{d}{dx}\frac{\partial G}{\partial V'} = \frac{V\cos\gamma\, C_T \frac{d}{dx}\frac{\partial F}{\partial V'} - C_T\left(\frac{\partial F}{\partial V'}\right)(V'\cos\gamma - V\cdot\gamma'\sin\gamma)}{(V\cos\gamma)^2}$$

Using an abbreviated notation: $\frac{\partial Y}{\partial Z} = Y_Z$

$$\frac{d}{dx}\left(\frac{\partial G}{\partial V'}\right) = G\left[\frac{F_{V'}}{F} - \frac{F_{V'}\ (V'\cos\gamma - V\gamma'\sin\gamma)}{F V\cos\gamma}\right]$$

or

$$G_{V'}{}' = G\left[\frac{F_{V'}{}'}{F} - \frac{V' F_{V'}}{V\,F} + \frac{F_{V'}}{F}\,\gamma'\tan\gamma\right]$$

$$= G\,\frac{F_{V'}}{F}\left[\frac{F_{V'}'}{F_{V'}} - \frac{V'}{V} + \gamma'\tan\gamma\right]$$

Now:

$$G_V - G_{V'}' = 0 = G\left[\frac{F_V}{F} - \frac{1}{V}\right] - G\,\frac{F_{V'}}{F}\left[\frac{F_{V'}'}{F_{V'}} - \frac{V'}{V} + \gamma'\tan\gamma\right]$$

Multiplying through by $\frac{F}{G}$ gives us the first Euler

$$F_V - \frac{F}{V} - F_{V'}' + \frac{V'}{V}F_{V'} - F_{V'}\gamma'\tan\gamma = 0 \qquad (34)$$

2.) Derivation of the Second Euler equation

$$\frac{\partial G}{\partial\gamma} - \frac{d}{dx}\frac{\partial G}{\partial\gamma'} = 0$$

as before:

$$G = \frac{C_T F}{V\cos\gamma}$$

$$\frac{\partial G}{\partial\gamma} = \frac{V\cos\gamma\, C_T\left(\frac{\partial F}{\partial\gamma}\right) + C_T F V\sin\gamma}{(V\cos\gamma)^2}$$

$$= G\left[\frac{F_\gamma}{F} + \tan\gamma\right]$$

and

$$\frac{\partial G}{\partial\gamma'} = \frac{C_T}{V\cos\gamma}\left(\frac{\partial F}{\partial\gamma'}\right)$$

then

$$\frac{d}{dx}\left(\frac{\partial G}{\partial\gamma'}\right) = C_T\left[\frac{V\cos\gamma\,\frac{dF_{\gamma'}}{dx} - F_{\gamma'}(V'\cos\gamma - V\gamma'\sin\gamma)}{(V\cos\gamma)^2}\right]$$

$$\frac{d}{dx}\frac{\partial G}{\partial\gamma'} = G\left[\frac{F_{\gamma'}'}{F} - \frac{F_{\gamma'}}{F}\left(\frac{V'}{V} - \gamma'\tan\gamma\right)\right]$$

$$= G\,\frac{F_{\gamma'}}{F}\left[\frac{F_{\gamma'}'}{F_{\gamma'}} - \frac{V'}{V} + \gamma'\tan\gamma\right]$$

Now forming the Euler:

$$G_\gamma - G_{\gamma'}{}' = 0 = G\left[\frac{F_\gamma}{F} + \tan\gamma\right] - \frac{G\,F_{\gamma'}}{F}\left[\frac{F_{\gamma'}{}'}{F_{\gamma'}} - \frac{V'}{V} + \gamma'\tan\gamma\right]$$

Multiplying through by F/G leaves the second equation:

$$F_\gamma + F\tan\gamma - F_{\gamma'}{}' + V'\,F_{\gamma'} - F_{\gamma'}\,\gamma'\tan\gamma = 0 \qquad (35)$$

3.) Derivation of Third Euler

$$\frac{\partial G}{\partial W} - \frac{d}{dx}\left(\frac{\partial G}{\partial W'}\right) = 0$$

$$G = \frac{C_T\,F}{V\cos\gamma}$$

$$\frac{\partial G}{\partial W} = \frac{C_T}{V\cos\gamma}\left(\frac{\partial F}{\partial W}\right)$$

$$\frac{\partial G}{\partial W'} = \frac{\partial W'}{\partial W'} = 1$$

$$\frac{d}{dx}\left(\frac{\partial G}{\partial W'}\right) = 0$$

and

$$G_W - G_{W'}{}' = G\left[\frac{F_W}{F}\right] = 0 \qquad (36)$$

Therefore $F_W = 0$ and there is no Third Euler in this case.

b) Maximum Range Equations (i.e. W is independent variable)

Note: $\frac{d(\ )}{dW} \equiv (\ )'$

From equation (15)

$$G = -\frac{V\cos\gamma}{C_T\,F}$$

1.) Derivation of the First Euler Equation

$$\frac{\partial G}{\partial V} = \frac{\cos\gamma}{C_T}\left[\frac{-F + V\left(\frac{\partial F}{\partial V}\right)}{F^2}\right]$$

$$= G\left[\frac{1}{V} - \frac{F_V}{F}\right]$$

$$\frac{\partial G}{\partial V'} = -\frac{V\cos\gamma}{C_T}\left[-\frac{\partial F/\partial V'}{(F)^2}\right]$$

$$= \frac{V\cos\gamma \; F_{V'}}{C_T \; F^2}$$

$$\frac{d}{dW}\left(\frac{\partial G}{\partial V'}\right) = \frac{C_T \; F^2}{(C_T \; F^2)^2}\left(V' \cos\gamma \, F_{V'} + V\cos\gamma \, F_{V'}{}' - V\gamma'\sin\gamma \, F_{V'}\right)$$

$$-\left(\frac{V\cos\gamma \, F_{V'} \; 2F \, F' \, C_T}{(C_T \; F^2)^2}\right)$$

$$= \frac{V' \, F_{V'} \cos\gamma}{C_T \; F^2} + \frac{V\cos\gamma \, F_{V'}{}'}{C_T \; F^2} - \frac{F_{V'} \, V \gamma'\sin\gamma}{C_T \; F^2}$$

$$-\frac{2V\cos\gamma \; F_{V'} \, F'}{C_T \; F^3}$$

$$G_{V'}{}' = -G\left[\frac{V'}{V}\frac{F_{V'}}{F} + \frac{F_{V'}{}'}{F} - \frac{\gamma'\tan\gamma}{F} - \frac{2F_{V'}F'}{F^2}\right]$$

$$= G\left[\frac{F_{V'}}{F}\right]\left[\frac{2F'}{F} + \gamma'\tan\gamma - \frac{F_{V'}{}'}{F_{V'}} - \frac{V'}{V}\right]$$

Forming the Euler

$$G_V - G_{V'} = G\left[\frac{1}{V} - \frac{F_V}{F}\right] - \frac{G \, F_{V'}}{F}\left[\frac{2F'}{F} + \gamma'\tan\gamma - \frac{F_{V'}{}'}{F_{V'}} - \frac{V'}{V}\right] = 0$$

And

$$\frac{F}{V} - F_V - \frac{2F_{V'} \, F'}{F} - F_{V'} \, \gamma'\tan\gamma + F_{V'}{}' + \frac{V' \, F_{V'}}{V} = 0 \qquad (37)$$

2.) Derivation of the Second Euler Equation

$$\frac{\partial G}{\partial \gamma} - \frac{d}{dW}\left(\frac{\partial G}{\partial \gamma'}\right) = 0$$

$$\frac{\partial G}{\partial \gamma} = \frac{V}{C_T}\left[\frac{-F\sin\gamma - \cos\gamma\,(\partial F/\partial \gamma)}{F^2}\right]$$

$$= -G\left[\tan\gamma + \frac{F_\gamma}{F}\right]$$

$$\frac{\partial G}{\partial \gamma'} = \frac{V\cos\gamma}{C_T}\left[-\frac{\partial F/\partial \gamma'}{(F)^2}\right] = \frac{-VF_{\gamma'}\cos\gamma}{C_T F^2}$$

$$\frac{d}{dW}\left(\frac{\partial G}{\partial \gamma'}\right) = -\frac{1}{C_T}\left[\frac{F_{\gamma'}\cos\gamma\, V'}{F2} + \frac{V\cos\gamma\, F_{\gamma'}{}'}{F2} - \frac{V F_{\gamma'}\,\gamma'\sin\gamma}{F2}\right.$$

$$\left. - \frac{V F_{\gamma'}\cos\gamma\,(2F\,F')}{F^4}\right]$$

$$\frac{d}{dW}\frac{\partial G}{\partial \gamma'} = G_{\gamma'}{}' = -G\left[\frac{V'}{V}\frac{F_{\gamma'}}{F} + \frac{F_{\gamma'}{}'}{F} - \frac{\gamma' F_{\gamma'}\tan\gamma}{F} - \frac{2F_{\gamma'}F'}{F2}\right]$$

Now

$$G_\gamma - G_{\gamma'}{}' = -G\left[\tan\gamma + \frac{F_\gamma}{F}\right] + G\left[\frac{V'}{V}\frac{F_{\gamma'}}{F} + \frac{F_{\gamma'}{}'}{F} - \frac{\gamma' F_{\gamma'}\tan\gamma}{F}\right.$$

$$\left. - \frac{2F_{\gamma'}F'}{F2}\right] = 0$$

From which:

$$F\tan\gamma + F_\gamma - \frac{V'}{V}F_{\gamma'} - F_{\gamma'}{}' + \gamma' F_{\gamma'}\tan\gamma + \frac{2F_{\gamma'}F'}{F} = 0 \qquad (38)$$

Thus equations (34), (35), (36), (37), and (38) represent extremal paths in terms of flight path angle (γ), velocity (V), horizontal distance (x) and pounds of fuel (W). The first three equations represent extremal paths for minimum (or maximum) fuel consumption for

traveling a given distance x, and the last two equations represent an extremal path for maximum (or minimum) distance x for a given fuel expenditure W.

PART III

THE BOUNDED CASE

The equations derived in section D of Part II apply to a more or less general propelled body with unlimited thrust and thus unlimited velocity and flight path angle. They do not consider any boundaries on the thrust available to follow the extremal path that they designate. This extremal path may very well call for altitudes, velocities, and accelerations above the physical ability of a particular airframe - engine combination for all or part of the path. Those equations are, therefore, the absolute extremums for an ideal body without limitations imposed on any of its parameters. They define a path that is the best possible and should be followed whenever that path is within the operating capabilites of the particular propelled body.

The primary limitation imposed by physical factors in any particular case is that of thrust. Thrust varies from zero to a finite maximum value that is a function of the individual thrust producer and other variables. For a given type power plant the maximum effective thrust is usually a function of the variables of the problem. For example: The thrust of a rocket is a constant; the thrust of a ram jet varies with velocity and altitude; the thrust of a turbojet varies with velocity and altitude. Therefore, for any given airframe - engine combination there is a limiting envelope or boundary of thrust which limits the path variables such as velocity, flight path angle, altitude, etc. Whenever the general path equations of

Part II demand a thrust that exceeds this boundary, the path becomes unrealistic and of academic interest only. It then becomes necessary to compute a new path from this point that will be an extremal based upon a maximum continuous value of thrust available. Euler equations will now be derived to define an extremal path for the case of bounded thrust.

A.) The Bounded Case for Minimum Fuel for Turbojets

As was said before, the thrust of a turbojet varies with altitude and velocity, i.e.:

$F = f(V,h)$ where f pertains to bounded thrust

But since altitude is a function of path angle, we may also say:

$$F = f(V,\gamma)$$

From before:

$$G = \frac{C_T F}{V\cos\gamma}$$

But now

$$G = \frac{C_T f}{V\cos\gamma} \qquad (39)$$

This function is independent of the characteristics of the airframe - engine combination and must be modified to include them. From (21b):

$$V\cos\gamma = \frac{g(f - D - W\sin\gamma)}{WV'}$$

Substituting this into (39) above

$$G = \frac{C_T f W V'}{g(f-D-W\sin\gamma)} \qquad (40)$$

1.) Derivation of the First Euler Equation

$$\frac{\partial G}{\partial V} - \frac{d}{dx}\left(\frac{\partial G}{\partial V'}\right) = 0$$

$$\frac{\partial G}{\partial V} = G\left[\frac{\partial f/\partial V}{f} - \frac{\partial f/\partial V - \partial D/\partial V}{f - D - W\sin\gamma}\right] \qquad (41)$$

$$\frac{\partial G}{\partial V'} = \frac{\sigma_T\, f\, W}{g(f - D - W\sin\gamma)} = \frac{G}{V'}$$

But

$$G = W'$$

Then

$$\frac{\partial G}{\partial V'} = \frac{W'}{V'}$$

Now

$$\frac{d}{dx}\left(\frac{\partial G}{\partial V'}\right) = \frac{V'W'' - W'V''}{(V')^2} \qquad (42)$$

Forming the Euler (41) - (42) = 0

$$\frac{\partial G}{\partial V} - \frac{d}{dx}\left(\frac{\partial G}{\partial V'}\right) = W'\left[\frac{f_V}{f} - \frac{f_V - D_V}{(f-D-W\sin\gamma)}\right] - \frac{V'W'' - W'V''}{(V')^2} = 0$$

From which

$$\frac{W''}{W'} - V'\left[\frac{f_V}{f} - \frac{f_V - D_V}{(f-D-W\sin\gamma)}\right] - \frac{V''}{V'} = 0 \qquad (43)$$

2.) Derivation of Second Euler Equation

$$\frac{\partial G}{\partial \gamma} = W'\left[\frac{\partial f/\partial\gamma}{f} - \frac{\partial f/\partial\gamma - \partial D/\partial\gamma - W\cos\gamma}{(f - D - W\sin\gamma)}\right] \qquad (44)$$

$$\frac{\partial G}{\partial \gamma'} = W'\left[\frac{\partial D/\partial\gamma'}{(f-D-W\sin\gamma)}\right]$$

$$\frac{d}{dx}\left(\frac{\partial G}{\partial \gamma'}\right) = W'\left[\frac{(f-D-W\sin\gamma)D_{\gamma'}'}{(f-D-W\sin\gamma)^2} - D_{\gamma'}\frac{(f'-D'-W'\sin\gamma - W\gamma'\cos\gamma)}{(f-D-W\sin\gamma)^2}\right]$$

$$+ W''\left[\frac{D_{\gamma'}}{(f-D-W\sin\gamma)}\right] \qquad (45)$$

Forming the Euler (44) - (45) = 0

$$W''\left[\frac{D_{\gamma'}}{(f-D-W\sin\gamma)}\right] - W'\left[\frac{f_\gamma}{f} - \frac{(f_\gamma - D_\gamma - W\cos\gamma)}{(f - D-W\sin\gamma)}\right]$$

$$+ \; W'\left[\frac{D_{\gamma'}'}{(f-D-W\sin\gamma)} - D_{\gamma'}\frac{(f'-D'-W'\sin\gamma - W\gamma'\cos\gamma)}{(f-D-W\sin\gamma)^2}\right] = 0$$

From which

$$\frac{W''}{W'} D_{\gamma'} - \frac{f_\gamma}{f}(f-D-W\sin\gamma) + f_\gamma - D_\gamma - W\cos\gamma + D_{\gamma'}'$$

$$- \; D_{\gamma'}\frac{(f'-D'-W'\sin\gamma - W\gamma'\cos\gamma)}{(f-D-W\sin\gamma)} = 0 \qquad (46)$$

Partial derivatives of drag with respect to V' and thrust with respect to γ' are zero since drag and thrust are not functions of V' and γ' respectively.

E) The Bounded Case for Maximum Range (i.e. with W as independent variable)

As before

$$G = \frac{V\cos\gamma}{C_T F}$$

and

$$F = f(V,h) = f(V,\gamma) \text{ only}$$

so

$$G = \frac{V\cos\gamma}{C_T f} \qquad (47)$$

Introducing the airframe characteristeics:

From (13a)

$$L + \frac{W}{g} V \gamma' C_T F - W\cos\gamma = 0$$

$$\cos\gamma = \frac{L}{W} + \frac{V \gamma' C_T f}{g}$$

Substituting this into (47) leaves:

$$G = \frac{VL}{C_T W f} + \frac{V^2 \gamma'}{g} \qquad (48)$$

1.) Derivation of First Euler Equation

$$\frac{\partial G}{\partial V} = \frac{L}{C_T W f} + \frac{V L_V}{C_T W f} - \frac{V L_V f}{C_T W f^2} + \frac{2 V \gamma'}{g} \qquad (49)$$

$$\frac{\partial G}{\partial V'} = \frac{V L_{V'}}{C_T W f} \quad \text{(thrust is not a function of } V'\text{)}$$

$$\frac{d}{dW}\left(\frac{\partial G}{\partial V'}\right) = \frac{V L_{V'}' + V' L_{V'}}{C_T W f} - \frac{V L_{V'} (W f' + f)}{C_T (W f)^2} \qquad (50)$$

Forming the Euler (49) - (50) = 0

$$L + V L_V - V L \frac{f_V}{f} + 2 C_T \frac{W}{g} \gamma' f V - V L_{V'}' - V' L_{V'} + \frac{V L_{V'}}{W f}\left[W f' + f\right] = 0 \qquad (51)$$

2.) Second Euler Equation

$$\frac{\partial G}{\partial \gamma} = \frac{V}{C_T W f}\left[L_\gamma - \frac{L f_\gamma}{f}\right] \qquad (52)$$

$$\frac{\partial G}{\partial \gamma'} = \frac{V L_{\gamma'}}{C_T W f} + \frac{V^2}{g}$$

$$\frac{d}{dx}\left(\frac{\partial G}{\partial \gamma'}\right) = \frac{V\,L_{\gamma'}'}{C_T W f} + \frac{V'\,L_{\gamma'}}{C_T W f} - \frac{V L_{\gamma'}\left[W f' + f\right]}{C_T (W f)^2} + \frac{2 V V'}{g} \tag{53}$$

Forming the Euler (52) - (53) = 0

$$L_\gamma - L\,\frac{f_\gamma}{f} - L_{\gamma'}' - \frac{V'}{V}\,L_{\gamma'} + \frac{L_{\gamma'}}{W f}\left[W f' + f\right] - \frac{2 C_T W}{g}\,f\,V' = 0 \tag{54}$$

PART IV: EVALUATION OF THE DERIVATIVES AND FORMULATION OF THE FINAL EQUATIONS

The Euler equations as developed in Parts II and III contain derivatives on several of the variables. To determine the desired path, these derivatives must be evaluated and included or discarded depending upon their effect. In this thesis only the derivatives for the minimum fuel path, both bounded and unbounded, will be determined. The evaluation of the derivatives for the maximum range path (W independant variable) is analogous but the expressions are much more unwieldy.

Considering first the derivatives to be evaluated for the first Euler in the unbounded case, we have: (1)F_V, (2)$F_{V'}$, (3)$F_{V'}'$. From equation (21b)

$$F = D + \frac{W}{g} V' V\cos\gamma + W\sin\gamma \tag{55}$$

Taking the desired derivatives:

$$F_V = D_V + \frac{W}{g} V' \cos\gamma \tag{56}$$

$$F_{V'} = D_{V'} + \frac{W}{g} V\cos\gamma \tag{57}$$

$$F_{V'}' = D_{V'}' + \frac{W' V\cos\gamma}{g} + \frac{W}{g} V'\cos\gamma - \frac{\gamma' W V\sin\gamma}{g}$$

$$= D_{V'}' + \frac{W}{g} V\cos\gamma \left[\frac{W'}{W} + \frac{V'}{V} - \gamma'\tan\gamma \right] \tag{58}$$

Consider now the derivatives to be evaluated for the Euler in γ for the unbounded case; the following appear: (1)F_γ, (2)$F_{\gamma'}$, (3)$F_{\gamma'}'$. From (55)

$$F = D + \frac{W}{g} V V'\cos\gamma + W\sin\gamma$$

Whence:

$$F_{\gamma} = D_{\gamma} - \frac{W}{g} V V' \sin\gamma + W \cos\gamma \qquad (59)$$

$$F_{\gamma'} = D_{\gamma'} \qquad (60)$$

$$F_{\gamma'}' = D_{\gamma'}' \qquad (61)$$

These, then, are the desired derivatives as functions of drag or of drag derivatives. The drag derivatives will be determined after a look at the derivatives appearing in the bounded equations. From equations (43) and (46) for the bounded case we have the following derivatives:

(1) W'

(2) W''

(3) f_V

(4) f_{γ}

(5) f'

Now $$W' = \frac{C_T f W V'}{g(f - D - W\sin\gamma)} \qquad (62)$$

From which

$$W'' = \frac{C_T f W V'}{g(f - D - W\sin\gamma)} \left[\frac{f'}{f} + \frac{W'}{W} + \frac{V''}{V'} - \frac{(f' - D' - W'\sin\gamma - W\gamma'\cos\gamma)}{(f - D - W\sin\gamma)} \right] \qquad (63)$$

Since the thrust, f, used in the bounded equations is an actual thrust, (i.e. thrust available), engine data for the specific engine - airframe combination will supply the information necessary for evaluating the derivatives on thrust.

$f = f(V,h)$ A known relationship for a given aircraft

Then

$f_V = \partial f/\partial V$ This can be taken directly from engine curves (64a)

Now

$$f_\gamma = \frac{\partial f}{\partial h}\,\frac{\partial h}{\partial \gamma} \qquad (64b)$$

and $\partial f/\partial h$ is available from engine curves while $\partial h/\partial \gamma$ is a simple geometric relationship.

$$f' = \frac{\partial f}{\partial V}\,V' + \frac{\partial f}{\partial \gamma}\,\gamma' \qquad (64c)$$

To evaluate the derivatives on drag it is most convenient to express drag in the following form:

$$D = D_o + D_i$$

$$= \frac{1}{2}\rho S\, V^2 C_{D_o} + \frac{1}{2}\rho S\cdot V^2\,\frac{C_L{}^2}{\pi eAR}$$

Or, substituting for C_L its equivalent in terms of lift from equation (21a):

$$D = \frac{1}{2}\rho S\, V^2 C_{D_o} + \frac{2\,(W\cos\gamma)^2}{\pi eAR\,\rho S\, V^2}\left(1 + \frac{V^2\gamma}{g}\right)^2 \qquad (65)$$

In taking derivatives on the above equation for drag, the subsonic region only will be considered. No Mach effects will be present then and both C_{D_o} and e will be treated as constants. An extension of this analysis to include Mach effects will be found in Appendix B.

Thus

$$D_V = \rho SVC_{D_o} - \frac{4\,(W\cos\gamma)^2}{\pi eAR\,\rho S\, V^3}\left[1 - \left(\frac{\gamma' V^2}{g}\right)^2\right]$$

$$= \frac{2\,D_o}{V} - \frac{2\,D_i}{V}\;\frac{\left(1 - \frac{\gamma'\,V^2}{g}\right)}{\left(1 + \frac{\gamma'\,V^2}{g}\right)} \qquad (66)$$

$$D_{V'} = 0 \qquad (67)$$

$$D_{V''} = 0 \qquad (68)$$

$$D_\gamma = \frac{-4\,W\cos\gamma\sin\gamma}{\pi eAR\rho S\,V^2}\left(1 + \frac{V^2\gamma'}{g}\right)^2$$

$$= -2\,D_i\tan\gamma \tag{69}$$

$$D_{\gamma'} = \frac{4\,(W\cos\gamma)^2\left(1 + \frac{V^2\gamma'}{g}\right)}{\pi eAR\,\rho S\,V^2}\left(\frac{V^2}{g}\right)$$

$$= \frac{2\,D_i}{\left(\frac{g}{V^2} + \gamma'\right)} \tag{70}$$

$$D_{\gamma'}{}' = \frac{4W^2\cos^2\gamma\left(1 + \frac{\gamma' V^2}{g}\right)}{\pi eAR\,\rho S\,g}\left[\frac{2W'}{W} - 2\gamma'\tan\gamma + \frac{(V^2\gamma'' + 2\gamma' VV')}{g\left(1 + \frac{\gamma' V^2}{g}\right)} - \frac{\rho'}{\rho}\right]$$

$$= \frac{2D_i}{\left(\frac{g}{V^2} + \gamma'\right)}\left[\frac{2W'}{W} - 2\gamma'\tan\gamma - \frac{\rho'}{\rho} + \frac{V\,(V\gamma'' + 2V'\gamma')}{g\left(1 + \frac{\gamma' V^2}{g}\right)}\right] \tag{71}$$

$$D' = D_V V' + D_\gamma\gamma' + D_{\gamma'}\gamma'' + D_\rho\rho' + D_W W'$$

$$= 2D_o\left[\frac{V'}{V} + \frac{\rho'}{2\rho}\right] - 2D_i\left[\frac{\left(1 - \frac{\gamma' V^2}{g}\right)V'}{\left(1 + \frac{\gamma' V^2}{g}\right)V} + \gamma'\tan\gamma\right.$$

$$\left. - \frac{\gamma''}{\left(\frac{g}{V^2} + \gamma'\right)} - \frac{W'}{W} + \frac{\rho'}{2\rho}\right] \tag{72}$$

The atmosphere parameter, ρ, and its derivative, ρ', may be evaluated as follows:

For altitudes below 36,089 feet:

$$\rho = \rho_o(1 - 6.89 \times 10^{-6}\,h)^{4.256}$$

$$\rho' = -4.256\,\rho_o(1 - 6.89 \times 10^{-6}\,h)^{3.256}\left(6.89 \times 10^{-6}\,\frac{dh}{dx}\right)$$

But $\frac{dh}{dx} = \tan\gamma$

Then:

$$\frac{\rho'}{\rho} = -\frac{29.38\tan\gamma}{10^6 - 6.89h} \tag{73}$$

For altitudes above 36, 089 feet:

$$\rho = \rho_0(0.2971)\, e^{\frac{-(h-36089)}{20786}}$$

$$\rho' = \rho_0(0.2971)\, e^{\frac{-(h-36089)}{20786}} \left[\frac{\frac{-dh}{dx}}{20786}\right]$$

And

$$\rho'/\rho = \frac{-\tan\gamma}{20786} \qquad (74)$$

The formulation of the final equations for the unbounded case may now be made as follows:

$$F_V - \frac{F}{V} - F_{V'}{}' - F_{V'}\left(\gamma'\tan\gamma - \frac{V'}{V}\right) = 0$$

Substuting for F_V, F/V, $F_{V'}{}'$, and $F_{V'}$ from (55), (56), (57), and (58); and for D_V which occurs in F_V from (66) gives after rearranging:

$$\frac{D_o}{V} - \frac{D_i}{V}\left[\frac{\left(3 - \frac{\gamma' V}{g}\right)}{\left(1 + \frac{\gamma' V}{g}\right)}\right] - \frac{W}{g} V\cos\gamma\left[\frac{W'}{W} + \left(\frac{g}{V^2} + 2\gamma'\right)\tan\gamma\right] = 0$$

(75)

From (35), the second Euler equations:

$$F_\gamma + F\tan\gamma - F_{\gamma'}{}' - F_{\gamma'}\left(\gamma'\tan\gamma - \frac{V'}{V}\right) = 0$$

Substituting for F_γ, $F\tan\gamma$, $F_{\gamma'}{}'$ and $F_{\gamma'}$ from (59), (60) and (61) and for the drag derivatives which are introduced from (69), (70) and (71) gives after rearranging:

$$(D_o - D_i)\tan\gamma + \frac{W}{\cos\gamma} - \frac{2\,D_i}{\left(\frac{g}{V^2} + \gamma'\right)}\left[\frac{2W'}{W} - \gamma'\tan\gamma - \rho'/\rho - \frac{V'}{V} + \frac{V}{g}\,\frac{\gamma(V\gamma'' + 2V'\gamma')}{\left(1 + \frac{\gamma' V^2}{g}\right)}\right] \qquad (76)$$

Solving (75) for V' and (76) for γ'' yields the two equations whose solution will define the optimum path.

$$V' = \frac{g^2}{W C_T V^2 \cos\gamma}\left\{ D_0\left(1 - \frac{VC_T}{g}\right) - D_i\left[\frac{\left(3 - \frac{\gamma' V^2}{g}\right)}{\left(1 + \frac{\gamma' V^2}{g}\right)} + \frac{VC_T}{g}\right]\right\}$$

$$- \left[\left(\frac{g}{V}\right)^2 + \frac{gC_T}{V} + 2g\gamma'\right]\frac{\tan\gamma}{C_T} \qquad (77)$$

$$\gamma'' = \frac{\left(\frac{g}{V^2} + \gamma'\right)^2}{2D_i}\left[(D_o - D_i)\tan\gamma + \frac{W}{\cos\gamma}\right] - \left(\frac{g}{V2} + \gamma'\right)\left[\frac{2W'}{W}\right.$$

$$\left. - \frac{V'}{V} - \gamma'\tan\gamma - \frac{\rho'}{\rho}\right] - \frac{2V'\gamma'}{V} \qquad (78)$$

The final equations for the bounded case are formulated in like manner:

From (43), the first Euler equation

$$\frac{W''}{W'} - V'\left[\frac{f_V}{f} - \frac{f_V - D_V}{(f - D - W\sin\gamma)}\right] - \frac{V''}{V'} = 0$$

Substituting for W"/W' from (62) and (63) and for D_V and D' which then remain from (66) and (72) yields on rearranging:

$$2\,D_o\left[\frac{V'}{V} - \frac{1}{V} + \frac{\rho'}{2\rho}\right] + 2\,D_i\left[\frac{\left(1 - \frac{\gamma' V^2}{g}\right)}{\left(1 + \frac{\gamma' V^2}{g}\right)}\left(\frac{1}{V} - \frac{V'}{V}\right)\right.$$

$$\left. - \gamma'\tan\gamma + \frac{\gamma''}{\left(\frac{g}{V^2} + \gamma'\right)} + \frac{W'}{W} - \frac{\rho'}{2\rho}\right] - f' + W'\sin\gamma$$

$$+ W\gamma'\cos\gamma + \left(\frac{f'}{f} - \frac{V'f_V}{f} + \frac{W'}{W}\right)(f - D - W\sin\gamma) = 0 \qquad (79)$$

From (46), the second Euler equation:

$$\frac{W''}{W'}D_{\gamma}' + \frac{f_V}{f}(f - D - W\sin\gamma) - f_\gamma - D_\gamma - W\cos\gamma + D_{\gamma'}'$$

$$- D_{\gamma'} \frac{(f' - D' - W'\sin\gamma - W\gamma'\cos\gamma)}{(f - D - W\sin\gamma)} = 0$$

Substituting for W''/W' from (62) and (63); for $D_{\gamma'}$, $D_{\gamma'}'$, and D' from (70), (71), and (72) yields on rearranging:

$$\begin{aligned}
&\frac{f'}{f} + \frac{3W'}{W} + \frac{V''}{V'} - \frac{2(f' - W'\sin\gamma - W\gamma'\cos\gamma) - 2D'}{(f - D - W\sin\gamma)} \\
&+ \frac{\frac{g}{V^2} + \gamma'}{2D_i}\left[\frac{f_\gamma}{f}(f - D - W\sin\gamma) - f_\gamma + 2D_i \tan\gamma - W\cos\gamma\right] \\
&+ \frac{V(V\gamma'' + 2V'\gamma')}{\left(1 + \gamma'\frac{V^2}{g}\right)} - \frac{\rho'}{\rho} - 2\gamma'\tan\gamma = 0 \qquad (80)
\end{aligned}$$

Solving (79) for γ'' and (80) for V'' yields two equations whose solution will define the desired path:

$$\begin{aligned}
\gamma'' = &\left\{\left[\frac{1}{2D_i}\right]\left[f' - W'\sin\gamma - W\gamma'\cos\gamma - \left(\frac{f'}{f} - \frac{V'f_V}{f} + \frac{W'}{W}\right)(f - D - W\sin\gamma)\right.\right. \\
&\left.- 2D_o\left(\frac{V'}{V} - \frac{1}{V} + \frac{\rho'}{2\rho}\right)\right] - \left(\frac{1}{V} - \frac{V'}{V}\right)\frac{\left(1 - \frac{\gamma'V^2}{g}\right)}{\left(1 + \frac{\gamma'V^2}{g}\right)} \\
&\left.+ \gamma'\tan\gamma - \frac{W'}{W} + \frac{\rho'}{2\rho}\right\}\left\{\frac{g}{V^2} + \gamma'\right\} \qquad (81)
\end{aligned}$$

$$\begin{aligned}
V'' = &- \frac{V'f'}{f} - \frac{3W'V'}{W} + \frac{2V'(f' - W'\sin\gamma - W\gamma'\cos\gamma) - 2V'D'}{(f - D - W\sin\gamma)} \\
&- \frac{V'\left(\frac{g}{V^2} + \gamma'\right)}{2D_i}\left[\frac{f_\gamma}{f}(f - D - W\sin\gamma) - f_\gamma + 2D_i\tan\gamma - W\cos\gamma\right] \\
&- VV'\frac{(V\gamma'' + 2V'\gamma')}{\left(1 + \frac{\gamma'V^2}{g}\right)} + \frac{V'\rho'}{\rho} + 2V'\gamma'\tan\gamma \qquad (82)
\end{aligned}$$

PART V: METHOD OF SOLUTION OF THE EQUATIONS FOR AN EXTREMAL PATH

Considering first the unbounded or general equations for an aircraft flying a minimum fuel path, the following equations control the aircraft's motion: (21b), (65), (77) and (78). These equations will be repeated here for ready reference:

A. The general equation of motion along the flight path:

$$F = D + \frac{W}{g} V' \quad V\cos\gamma + W\sin\gamma \tag{A}$$

B. The drag equation:

$$D = \frac{1}{2}\rho S V^2 C_{D_o} + \frac{2(W\cos\gamma)^2}{\pi e AR \rho S V^2}\left(1 + \frac{V^2\gamma'}{g}\right)^2 \tag{B}$$

C. The first Euler equation:

$$V' = \frac{g}{W C_T V^2 \cos\gamma}\left\{ D_o\left(1 - \frac{V C_T}{g}\right) - D_i\left[\frac{\left(3 - \frac{\gamma' V^2}{g}\right)}{\left(1 + \frac{\gamma' V^2}{g}\right)} + \frac{V C_T}{g}\right]\right\}$$

$$- \left[\left(\frac{g}{V}\right)^2 + \frac{g C_T}{V} + 2g\gamma'\right] \frac{\tan\gamma}{C_T} \tag{C}$$

D. The second Euler equation:

$$\gamma'' = \frac{\left(\frac{g}{V^2} + \gamma'\right)^2}{2 D_i}\left[(D_o - D_i)\tan\gamma + \frac{W}{\cos\gamma}\right] - \left(\frac{g}{V^2} + \gamma'\right)\left[\frac{2W'}{W}\right.$$

$$\left. - \frac{V'}{V} - \gamma'\tan\gamma - \frac{\rho'}{\rho}\right] - \frac{2V'\gamma'}{V} \tag{D}$$

where

$$D_o = \frac{1}{2}\rho S V^2 C_{D_o}$$

$$D_i = \frac{2(W\cos\gamma)^2}{\pi e AR \rho S V^2}\left(1 + \frac{\gamma' V^2}{g}\right)^2$$

This system is composed of four nonlinear differential equations

in γ and V. Three of the equations are first order and one is second order. This is equivalent to a third order equation in one unknown and thus three initial or boundary conditions may be specified.

Since no general closed form solution exists to this system, we are forced to integrate by means of a step-by-step process. This requires the use of initial conditions rather than boundary conditions. From equations (A) and (B) it can be seen that the forces on the airplane depend only on V, V', γ, and γ' and therefore our initial values should be given in terms of three of these at a given altitude, h. Since V' can also be determined directly in terms of the other three from equation (C), it will be most convenient to specify V, γ, and γ' initially.

For a given aircraft we know W_o and C_{D_o}, and for the unbounded case we assume unlimited thrust and knowledge of C_T at any thrust. Now, if we specify V, γ, and γ' initially, V' is computed from (C) and D, D_o and D_i from (B). With V' and D known, F is determined from (A). Since C_T is a function of F, it was necessary to estimate initially to find V'. Now that F has been determined, an iterative procedure should be used to correct V', C_T and F. W' is found easily from:

$$W' = -\frac{C_T F}{V\cos\gamma} \tag{E}$$

Now γ'' can be computed from equation (D). Thus we have defined all quantities for our starting point by initially specifying three variables V, γ, and γ'.

By using the first two terms of a simple Taylor series expansion we may now integrate step-wise to point number 1 and so on in the following manner:

Knowing V_o', γ_o', W_o' and γ_o'' we compute by Taylor series

$$V_1 = V_o + V_o' \Delta x$$
$$\gamma_1 = \gamma_o + \gamma_o' \Delta x$$
$$\gamma_1' = \gamma_o' + \gamma_o'' \Delta x \qquad \text{(F)}$$
$$W_1 = W_o + W_o' \Delta x$$

Also since

$$h' = \tan \gamma$$
$$h_1 = h_o + h' \Delta x \qquad \text{(F)}$$

Now with V_1, γ_1, γ_1', W_1, h_1 and equations (A) through (E) we can determine all quantities at point (1). Using Taylor series again we proceed to point (2) and so on.

If greater accuracy is required than obtained by two term Taylor series, additional terms of the series may be used, the Milne method may be used, or the increment in x may be decreased. Decreasing the increment in x reduces the errors due to non-linearities of the function involved. The Milne method is described in reference 2

In performing a step-by-step solution to the unbounded case, several considerations should be kept in mind to insure that the path so determined will be a "flyable" one:

1.) The equations can possibly give a maximum fuel path instead of a minimum.

2.) Flight path angle, γ, cannot be allowed to drive altitude to negative values or the a/c into stall.

3.) Accelerations and decelerations must be within the physical capabilities of the aircraft.

4.) Minor oscillations of the computed path in its initial stages are probably due to computational variables such as the size of x and to the initial conditions, and can be faired out for an actual flight path.

Juggling of initial conditions and increment in x will help avoid these difficulties at the beginning of the path.

To compute a minimum fuel path for the bounded case equations (C) and (D) must be replaced by (81) and (82). Equations (81) and (82) will be restated here only in functional form:

First Euler Equation

$$f(V, \gamma, V', \gamma', \gamma'') = 0 \tag{G}$$

Second Euler Equation

$$f(V, \gamma, V', \gamma', V'', \gamma'') = 0 \tag{H}$$

It must be remembered in the bounded case that thrust is fixed as a known function of V and γ for a particular aircraft. With this in mind, a close imspection of all of the equations for the bounded case (i.e. (A), (B), (G), (H)) tells us that the second Euler equation (H) is not necessary to the solution of the system. This is as we would suspect from the physical conditions of the bounded problem, since, by setting thrust at a fixed rating, say maximum continuous, it becomes a known function of V and γ. Thus one less equation is required for solving the sustem.

This can be seen more clearly from a look at (G) and (H), with the balance equations. We have two second order differential equations with second order terms on γ in both (G) and (H) equations and second order terms on V in (G) only. γ'' could be eliminated between (G) and (H) and our system would be equivalent to a third order equation in one unknown. Thus three initial conditions may be specified. Looking at (G) alone with the balance equations we also have an equivalent third order equation in one variable. Thus three initial conditions may be specified in either case.

If we initially specify V, γ, and γ' as before, for a particular aircraft, F, C_T, and D are determined. Then since F and D are known, V' is determined from equation (A). If V' is determined from equation (A), the V'' term in equation (H) is redundant. The other variables in equation (G) are functions of known or specified variables so V, γ, V', γ' and γ'' are related uniquely in (G) and therefore (H) can be disregarded. By using only the first Euler equation all quantities are specified or determined for the initial point of the bounded problem. By equations (F) representing the first two terms of the Taylor series we can again integrate stepwise to point 1 and so on as before.

The bounded problem and also the more general unbounded problem each have specific applications which fit completely within the scope of one or the other of the two situations. It is felt, however, in the absense of any computed results that the general solution will be a composite of the two. Due to the similarity of the systems of equations for the two cases, transition from one case to the other is

an easy matter at any point on the path. When thrust required exceeds that available while using the unbounded equations, transition to the bounded system of equations should be made. This transition should be accomplished at the last point on the path at which the thrust requirement is equal to the thrust available.

Now, while proceeding step-by-step along the path computed by the bounded system of equations, it is necessary that each path point be checked in the unbounded eqdations. At that point on the path where the unbounded thrust requirement comes within the envelope of thrust available from the power plant, transfer back to the unbounded system of equations should be made and computations continued until such time as the thrust requirement again forces transition to the bounded case. This procedure is continued until the required distance, x is covered.

CONCLUSIONS

It is concluded that a system of equations for maximum range or minimum fuel flight paths may be written which are not restricted to level flight or to negligible accelerations. The equations can be applied to flight paths which commence at take-off or at any other set of initial conditions from which an extremal path is desired.

It is further concluded that by specifying three initial conditions, methods of solution by stepwise integration exist for the system of equations.

It is also concluded that, should the extremal path call for greater thrust than is available from the aircraft, a maximum range or minimum fuel path can be computed based on maximum continuous thrust available. There are specific applications for both types of paths individually as well as for a composite path.

From Appendix A it is concluded that classical parameters for maximum range are restrictive applications of the more general equations derived in this thesis.

APPENDIX A

Reduction of Euler Equations to Classical Range Parameters

From the Euler equations representing a maximum range path at variable altitude it is a simple matter to derive equations or parameters for maximum range in level flight. Starting with equation (77)*

$$V' = \frac{g^2}{WC_T}\left\{\frac{\rho s C_{D_o}}{2\cos\gamma}\left(1 - \frac{C_T V}{g}\right) + \frac{2W^2\cos}{A\rho V^3}\left(1 + \frac{V^2\gamma'}{g}\right)\left[\frac{V\gamma'}{g} - \frac{3}{V} - \frac{C_T}{g}\left(1 + \frac{V^2\gamma'}{g}\right)\right] - \frac{W\tan\gamma}{V^2}\left(1 + \frac{C_T V}{g}\right)\right\}$$

Imposing the conditions for level flight ($\gamma = \gamma' = 0$) we have remaining:

$$V' = \frac{g^2}{WC_T}\left\{\frac{\rho s C_{D_o}}{2}\left(1 - \frac{C_T V}{g}\right) + \frac{2W^2}{A\rho V^3}\left[-\frac{3}{V} - \frac{C_T}{g}\right]\right\} \qquad (A.1)$$

Substituting for W^2 its level flight equivalent ($W = \frac{1}{2}\rho s V^2 C_L$)

$$\frac{2W^2}{\pi e AR\, s\rho V^3} = \frac{1}{2}\rho s V\frac{C_L{}^2}{\pi e AR} = \frac{1}{2}\rho s V C_{D_i}$$

Then

$$\frac{V'WC_T}{g^2} = \frac{\rho s C_{D_o}}{2}\left(1 - \frac{C_T V}{g}\right) + \frac{1}{2}\rho s V C_{D_i}\left(-\frac{3}{V} - \frac{C_T}{g}\right)$$

And

$$\frac{V'C_L C_T V^2}{g^2} = C_{D_o}\left(1 - \frac{C_T V}{g}\right) - C_{D_i}\left(3 + \frac{C_T V}{g}\right) \qquad (A.2)$$

Assuming nominal values of C_T, V, and C_L, say:

$C_T = 1$ lb/lb hr $= 1/3600$ sec

$V = 400$ ft/sec

$C_L = .2$

---

* in slightly different form

Then $\frac{C_T V}{g} \doteq .0035$ and may be considered negligible with respect to 1.

And $\frac{V' C_L C_T V^2}{g^2} \doteq .009 V'$ where V' is in $\frac{ft/sec}{ft}$ and is an extremely small quantity. This term may be considered to be zero. With these assumptions the equation reduces to:

$$C_{D_o} = 3 C_{D_i} = 3 C_D - 3 C_{D_o}$$

Or:

$$C_D = \frac{4}{3} C_{D_o} \qquad (A.3)$$

Equation (A.3) represents a parameter defining the speed schedule for maximum range at any one given altitude. This parameter is perhaps not as familiar as the classical

$$C_L^{1/2}/C_D = K \qquad (A.4)$$

parameter but it can be shown that the two are equivalent.

Differentiating $C_L^{1/2}/C_D = K$

$$\frac{1}{2} C_L^{-1/2} = K \frac{dC_D}{dC_L} = K \frac{2C_L}{\pi e AR}$$

Then

$$K = \frac{\pi e AR}{4C_L^{3/2}}$$

and

$$\frac{C_L^{1/2}}{C_D} = \frac{\pi e AR}{4C_L^{3/2}}$$

from which

$$C_D = \frac{4\,C_L^2}{\pi e AR} = 4\,C_{D_i} = 4\,C_D - 4C_{D_o}$$

and

$$C_D = \frac{4}{3} C_{D_o} \qquad (A.5)$$

APPENDIX B

Consideration of Mach Number Effects

In the foregoing developement of maximum range and minimum fuel equations, only the sub-sonic speed region was considered. In this section, Mach effects on the equations will be determined and applied, thereby removing any speed limitations.

Mach number appears explicitly in the drag terms of both the unbounded and bounded cases, and may possibly appear in the thrust, f, of the bounded case. Therefore, by determining the Mach corrections to the D and f terms and their derivatives, the complete system of equations is corrected and is applicable to the entire speed range.

Considering drag first, the Mach effects thereon may be accounted for as follows:

$$D = \frac{1}{2}\rho S V^2 C_D$$

$$= \frac{1}{2}\rho S V^2 C_{D_o} + \frac{2(W\cos\gamma)^2}{\pi e AR \rho S V^2}\left(1 + \frac{V^2\gamma'}{g}\right)^2$$

Now C_{D_o} and e are functions of Mach number. Defining

$$C_{D_o} \equiv \eta(M)$$

$$e \equiv \xi(M)$$

Drag may be written as:

$$D = \frac{1}{2}\rho S V^2 \eta(M) + \frac{2(W\cos\gamma)^2\left(1 + \frac{V^2\gamma'}{g}\right)^2}{\pi AR \rho S V^2 \xi(M)}$$

Now

$$D_o = \frac{1}{2}\rho S V^2 \eta(M)$$

$$D_i = \frac{2(W\cos\gamma)^2\left(1 + \frac{V^2\gamma'}{g}\right)^2}{\pi AR \rho S V^2 \xi(M)}$$

Taking derivatives on drag:

$$\frac{\partial D}{\partial V} = \frac{\partial D}{\partial V} + \frac{\partial D}{\partial M}\frac{\partial M}{\partial V}$$

$$\frac{\partial D}{\partial M} = \frac{\partial D}{\partial \eta}\frac{\partial \eta}{\partial M} + \frac{\partial D}{\partial \xi}\frac{\partial \xi}{\partial M}$$

Since $M = V/V_c$

$$\frac{\partial M}{\partial V} = \frac{1}{V_c}$$

Therefore

$$\frac{\partial D}{\partial V} = \rho S V \eta(M) + \frac{4(W\cos\gamma)^2\left(1 + \frac{\gamma' V^2}{g}\right)\left(1 - \frac{\gamma' V^2}{g}\right)}{\pi AR\ S\rho V^3\ \xi(M)}$$

$$+ \left[\frac{1}{2}\rho S V^2 \frac{\partial \eta}{\partial M} - \frac{2(W\cos\gamma)^2\left(1 + \frac{V^2\gamma'}{g}\right)^2}{\pi AR\rho S\ V^2\ [\xi(M)]^2}\frac{\partial \xi}{\partial M}\right]\frac{1}{V_c}$$

Or

$$\frac{\partial D}{\partial V} = D_o\left[\frac{\partial \eta/\partial M}{V_c\,\eta(M)} + \frac{2}{V}\right] - D_i\left[\frac{\partial \xi/\partial M}{V_c\,\xi(M)} - \frac{2\left(1 - \frac{\gamma' V^2}{g}\right)}{V\left(1 + \frac{\gamma' V^2}{g}\right)}\right] \qquad \text{(B1)}$$

Since neither drag nor Mach number are functions of V', as before:

$$\frac{\partial D}{\partial V'} = 0\,, \quad \text{also since} \quad \frac{\partial M}{\partial \gamma} = \frac{\partial M}{\partial \gamma'} = 0$$

the derivatives on drag with respect to γ and γ' are unchanged from their subsonic form provided C_{D_o} and e are inserted as proper functions of Mach number.

To evaluate $D_{\gamma'}$, from (70) and the above discussion:

$$D_{\gamma'} = \frac{4(W\cos\gamma)^2\left(1 + \frac{\gamma' V^2}{g}\right)}{g\pi AR\rho S\ \xi(M)}$$

Now

$$D_{\gamma'}' = \frac{\partial D_{\gamma'}}{\partial M}\frac{\partial M}{\partial V} V' + \frac{\partial D_{\gamma'}}{\partial V} V' + \frac{\partial D_{\gamma'}}{\partial \gamma}\gamma' + \frac{\partial D_{\gamma'}}{\partial \gamma'}\gamma''$$

$$+ \frac{\partial D_{\gamma'}}{\partial \rho}\rho' + \frac{\partial D_{\gamma'}}{\partial W} W'$$

$$\frac{\partial D_{\gamma'}}{\partial M} = - \frac{4(W\cos\gamma)^2\left(1 + \frac{\gamma' V^2}{g}\right)}{g\pi AR\rho S\,[\xi(M)]^2}\frac{\partial \xi}{\partial M} \qquad \text{(B2)}$$

From (71) and (B2)

$$D_{\gamma'}' = \frac{2D_i}{\left(\frac{g}{V^2} + \gamma'\right)}\left[\frac{W'}{W} - 2\gamma'\tan\gamma - \frac{\rho'}{\rho} + \frac{V(V\gamma'' + 2V'\gamma')}{\left(1 + \frac{\gamma' V^2}{g}\right)}\right.$$

$$\left. - \frac{V'}{V_c\xi(M)}\frac{\partial \xi}{\partial M}\right]$$

To evaluate D'

$$D' = \frac{\partial D_V}{\partial M}\frac{\partial M}{\partial V} V' + \frac{\partial D}{\partial V} V' + \frac{\partial D}{\partial \gamma}\gamma' + \frac{\partial D}{\partial \gamma'}\gamma'' + \frac{\partial D}{\partial \rho}\rho' + \frac{\partial D}{\partial W} W'$$

Then from (B1) and (72)

$$D' = 2D_o\left[\left(\frac{\partial\eta/\partial M}{2V_c\eta(M)} + \frac{1}{V}\right)V' + \frac{\rho'}{2\rho}\right] - 2D_i\left\{\left[\frac{\partial\xi/\partial M}{2V_c\xi(M)}\right.\right.$$

$$\left.\left. - \frac{\left(1 - \frac{\gamma' V^2}{g}\right)}{V\left(1 + \frac{\gamma' V^2}{g}\right)}\right]V' + \gamma'\tan\gamma - \frac{\gamma''}{\left(\frac{g}{V^2} + \gamma'\right)} - \frac{W'}{W} + \frac{\rho'}{2\rho}\right\}$$

As was previously noted, the bounded thrust, f, may possibly be a function of Mach number. If this is the case, derivatives may be taken in the normal manner as follows:

$$f = f(V, h, M)$$

$$\frac{\partial f}{\partial V} = \frac{\partial f}{\partial M}\frac{\partial M}{\partial V} + \frac{\partial f}{\partial V}$$

$$= \frac{\partial f}{\partial M}\frac{1}{V_c} + \frac{\partial f}{\partial V}$$

And

$$f' = \frac{\partial f}{\partial M}\frac{\partial M}{\partial V} V' + \frac{\partial f}{\partial V} V' + \frac{\partial f}{\partial h}\frac{\partial h}{\partial \gamma} \gamma' + \text{------}$$

The partial derivatives with respect to the various variables being obtained from known power plant characteristics.

BIBLIOGRAPHY

1. Dommasch, D.O., *Analytic Developments of the Optimum Flight Path Equations,* Princeton University, Department of Aeronautical Engineering Report of March 16, 1955

*2. Bliss, G.A., *Calculus of Variations,* Open Court Publishing Co., LaSalle, Illinois, 1944

**3. Milne, W.E., *Numerical Calculus,* Princeton University Press, Princeton, New Jersey.

4. Dommasch, D.O., Sherby, S.S., and Connolly, T.F., *Airplane Aerodynamics,* Pitman Publishing Corporation, New York, Toronto, London.

5. Lippisch, A., *The Range of Jet Propelled Aircraft,* Cornell Aeronautical Laboratory, Buffalo, March 1947.

6. Savage, W.F., *The Determination of Constant Altitude Range Performance of Turbojet - Propelled Aircraft,* University of Kentucky, Engineering Experiment Station Bulletin, March 1950.

7. Perkins, C.D., and Hage, R.E., *Airplane Performance Stability and Control,* John Wiley and Sons, Inc, New York, New York.

* Reference 1

** Reference 2

Thesis
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